EFFICIENT REMEZ ALGORITHMS FOR THE DESIGN OF NONRECURSIVE FILTERS

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INTRODUCTION

Two methods have evolved for the design of nonrecursive (FIR) filters over the past 30 years or so:

- **The window method**
- **The weighted-Chebyshev method**

INTRODUCTION

Window Method

The window method uses the Fourier series in conjunction with a class of functions known as *window functions*.

Advantages

- **Closed-form method.**
- **It is easy to apply.**
- **The design entails a relatively insignificant amount of computation.**

Disadvantages

- **Designs are suboptimal.**
- **A higher-order filter is needed to satisfy the required specifications.**
- **A higher-order filter means more computations per sample, which implies that these filters are slower and less efficient in real-time applications.**

INTRODUCTION *Cont'd*

Weighted-Chebyshev Method

This is an iterative multi-variable optimization method based on the *Remez Exchange Algorithm*.

Advantages

- **Designs are optimal.**
- **Method is very flexible - can be used to design filters, differentiators, Hilbert transformers, etc.**
- **It yields equiripple solutions.**
- **Minimum filter order is achieve for the required specifications.**
- **Minimum filter order implies a more efficient and faster filter for real-time applications.**

Disadvantages

- **Their design requires a very large amount of computation.**
- **Not suitable for applications where the design has to be carried out in real- or quasi-real time, for example, in programable or adaptable filters.**

INTRODUCTION *Cont'd*

Objectives

- **The purpose of the lecture is the describe the basics of the weighted-Chebyshev method.**
- **Examine ways by which the efficiency of the design process can be improved and the amount of computation reduced.**
- **Suggest possible leads to further research on the subject.**

INTRODUCTION *Cont'd*

Historical Evolution

The development of the weighted-Chebyshev method is as follows:

- **Herrmann published a short paper in *Electronics Letters* in May 1970.**
- **Herrmann's contribution was followed soon after, in March 1971, by a paper by Hofstetter, Oppenheim, and Siegel.**
- **These contributions were followed by a series of papers, during the seventies, by Parks, McClellan, Rabiner, and Herrmann.**
- **These developments led, in turn, to the well-known McClellan-Parks-Rabiner computer program for the design of nonrecursive filters which has found widespread applications.**
- **The approach to weighted-Chebyshev filters to be presented is based on some of the papers published by McClellan, Parks, and Rabiner and includes several enhancements proposed by the speaker.**

NOTE: **See bibliography for details.**

PROBLEM FORMULATION

- **Consider a nonrecursive filter characterized by the transfer function**

$$H(z) = \sum_{n=0}^{N-1} h(nT) z^{-n}$$

and assume that

  - N **is odd,**
  - **the impulse response is symmetrical, and**
  - **the sampling frequency is $\omega_s = 2\pi$.**

- **Since $T = 2\pi/\omega_s = 1$ s, the frequency response of the filter can be expressed as**

$$H(e^{j\omega}) = e^{-jc\omega} P_c(\omega)$$

where

$$P_c(\omega) \;=\; \sum_{k=0}^{c} a_k \cos k\omega \tag{A}$$

is the *gain function* and

$$\begin{aligned} a_0 &= h(c) \\ a_k &= 2h(c-k) \quad \text{for } k = 1,\, 2,\, \ldots,\, c \\ c &= (N-1)/2 \end{aligned}$$

- **Note that $P_c(\omega)$ is the frequency response of a noncausal version of the required filter.**

ERROR FUNCTION

- **If $e^{-jc\omega}D(\omega)$ is the idealized frequency response of the desired filter and $W(\omega)$ is a weighting function, an error function $E(\omega)$ can be constructed as**

$$E(\omega) = W(\omega)[D(\omega) - P_c(\omega)]$$

where

$$P_c(\omega) = \sum_{k=0}^{c} a_k \cos k\omega$$

- **If $|E(\omega)|$ is minimized such that**

$$|E(\omega)| = |W(\omega)[D(\omega) - P_c(\omega)]| \leq \delta_p \quad \text{for } \omega \in \Omega \tag{B}$$

with respect to some compact (dense) subset of the frequency interval $[0,\, \pi]$, say Ω, a filter can be obtained in which

$$|E_0(\omega)| = |D(\omega) - P_c(\omega)| \leq \frac{\delta_p}{|W(\omega)|} \quad \text{for } \omega \in \Omega$$

LOWPASS FILTERS

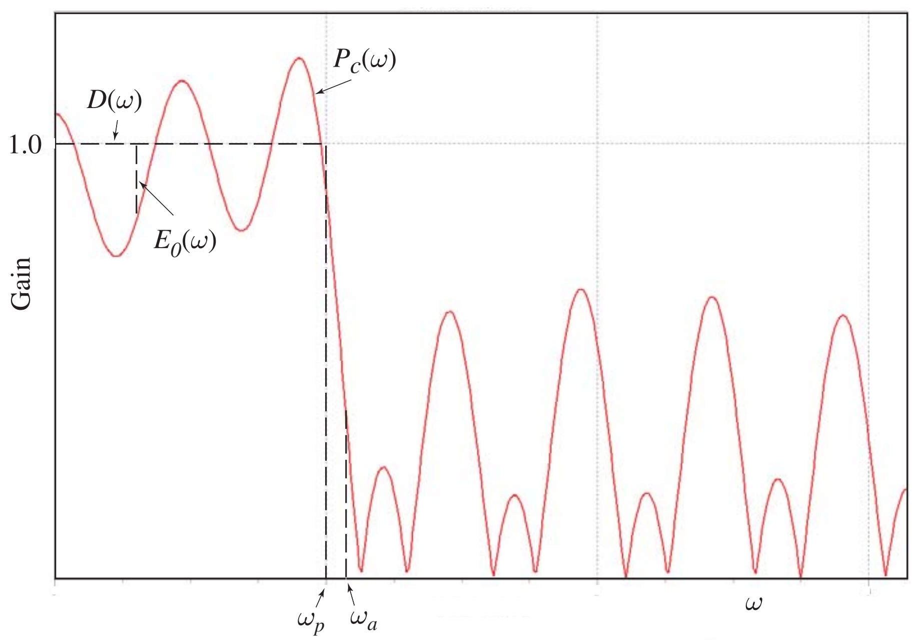
In the case of a lowpass filter, the minimization of $|E(\omega)|$ will force the inequality

$$|E_0(\omega)| = |D(\omega) - P_c(\omega)| \leq \frac{\delta_p}{|W(\omega)|} \quad \text{for } \omega \in \Omega \tag{C}$$

where

$$D(\omega) = \begin{cases} 1 & \textbf{for } 0 \leq \omega \leq \omega_p \\ 0 & \textbf{for } \omega_a \leq \omega \leq \pi \end{cases}$$

In effect, a minimization algorithm will force the actual gain function $P_c(\omega)$ to approach the ideal gain function $D(\omega)$ as depicted in the graph.



LOWPASS FILTERS *Cont'd*

- **If**

$$W(\omega) = \begin{cases} 1 & \textbf{for } 0 \le \omega \le \omega_p \\ \frac{\delta_p}{\delta_a} & \textbf{for } \omega_a \le \omega \le \pi \end{cases}$$

then from Eq. (C), i.e.,

$$|E_0(\omega)| = |D(\omega) - P_c(\omega)| \le \frac{\delta_p}{|W(\omega)|} \quad \text{for } \omega \in \Omega$$

we get

$$|E_0(\omega)| \le \begin{cases} \delta_p & \textbf{for } 0 \le \omega \le \omega_p \\ \delta_a & \textbf{for } \omega_a \le \omega \le \pi \end{cases}$$

- **Weighted-Chebyshev filters are so called because they have an *equiripple* amplitude response just like Chebyshev filters, as shown in the graph.**

There is no other relation between weighted-Chebyshev and

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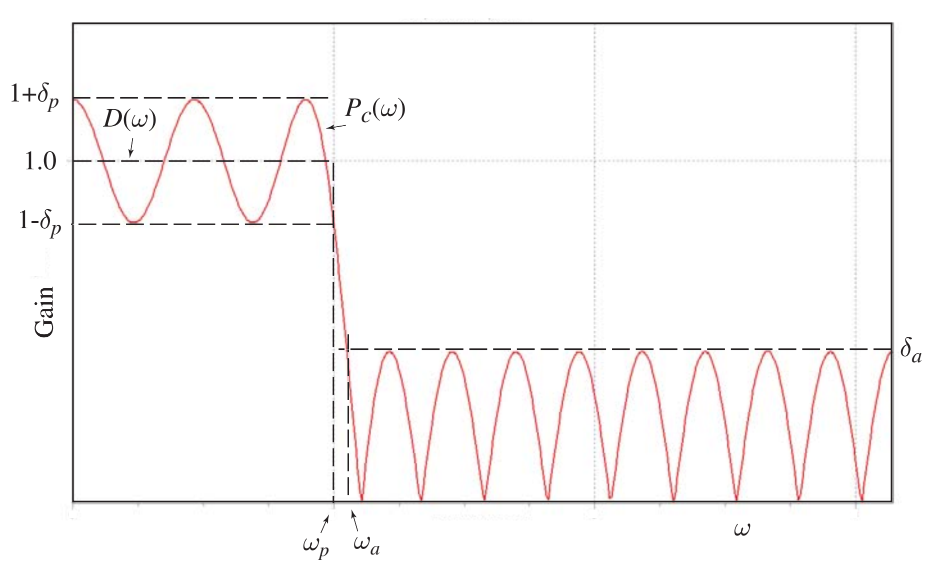


MINIMAX PROBLEM

The most appropriate approach for the solution of the optimization problem just described is to solve the minimax problem

$$\underset{\mathbf{x}}{\textbf{minimize}}\ \{\max_{\omega} |E(\omega)|\}$$

where

$$\mathbf{x} = [a_0 \quad a_1 \quad \cdots \quad a_c]^T$$

The solution of this problem exists by virtue of the so-called *alternation theorem*.

ALTERNATION THEOREM

If $P_c(\omega)$ is a linear combination of $r = c + 1$ cosine functions of the form

$$P_c(\omega) = \sum_{k=0}^{c} a_k \cos k\omega$$

then a necessary and sufficient condition that $P_c(\omega)$ be the unique, best, weighted-Chebyshev approximation to a continuous function $D(\omega)$ on Ω, where Ω is a compact (dense) subset of the frequency interval $[0,\ \pi]$, is that the weighted error function $E(\omega)$ exhibit at least $r + 1$ extremal frequencies in Ω, i.e., there must exist at least $r + 1$ points $\hat{\omega}_i$ in Ω such that

$$\hat{\omega}_0 < \hat{\omega}_1 < \cdots < \hat{\omega}_r$$

$$E(\hat{\omega}_i) = -E(\hat{\omega}_{i+1}) \quad \text{for } i = 0,\ 1,\ \ldots,\ r - 1$$

and

$$|E(\hat{\omega}_i)| = \max_{\omega \in \Omega} |E(\omega)| \quad \text{for } i = 0,\ 1,\ \ldots,\ r$$

ALTERNATION THEOREM *Cont'd*

- **From the alternation theorem and Eq. (B), i.e.,**

$$E(\omega) = W(\omega)[D(\omega) - P_c(\omega)]$$

we can write

$$E(\hat{\omega}_i)) = W(\hat{\omega}_i))[D(\hat{\omega}_i)) - P_c(\hat{\omega}_i))] = (-1)^i\delta$$

for $i == 0,\ 1,\ \ldots,\ r$**, where** δ **is a constant.**

- **The above system of equations can be put in matrix form as**

$$\begin{bmatrix} 1 & \cos\hat{\omega}_0 & \cos\hat{\omega}_0 & \cdots & \cos\hat{\omega}_0 & \frac{1}{W(\hat{\omega}_0)} \\ 1 & \cos\hat{\omega}_1 & \cos\hat{\omega}_1 & \cdots & \cos\hat{\omega}_1 & \frac{1}{W(\hat{\omega}_1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \cos\hat{\omega}_r & \cos\hat{\omega}_r & \cdots & \cos\hat{\omega}_r & \frac{1}{W(\hat{\omega}_r)} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_c \\ \delta \end{bmatrix} = \begin{bmatrix} D(\hat{\omega}_0) \\ D(\hat{\omega}_1) \\ \vdots \\ D(\hat{\omega}_{r-1}) \\ D(\hat{\omega}_r) \end{bmatrix}$$

- **If the extremal frequencies (or extremals for short) were known, coefficients** a_k **and, in turn, the frequency response of the filter could be computed using Eq. (A), i.e.,**

$$P_c(\omega) = \sum_{k=0}^{c} a_k \cos k\omega$$

- **The solution of this system exists since the above** $(r+1) \times (r+1)$ **matrix can be shown to be nonsingular.**

REMEZ EXCHANGE ALGORITHM

The Remez exchange algorithm is an *iterative multivariable algorithm* which is naturally suited for the solution of the minimax problem just described.

It is based on the second optimization method of Remez.

(See bibliography.)

BASIC REMEZ EXCHANGE ALGORITHM

1. **Initialize extremal frequencies $\hat{\omega}_0, \hat{\omega}_1, \ldots, \hat{\omega}_r$ and ensure that an extremal is assigned at each band edge.**

2. **Solve the system of equations to get δ and the coefficients $a_0, a_1, \ldots, a_c$.**

3. **Using the coefficients $a_0, a_1, \ldots, a_c$, calculate $P_c(\omega)$ and the magnitude of the error**

$$|E(\omega)| = |W(\omega)[D(\omega) - P_c(\omega)]|$$

4. **Locate the frequencies $\overset{\cap}{\omega}_0, \overset{\cap}{\omega}_1, \ldots, \overset{\cap}{\omega}_\rho$ at which $|E(\omega)|$ is maximum and $|E(\overset{\cap}{\omega}_i)| \geq \delta$.**

   (These frequencies are *potential extremals* for the next iteration.)

5. **Compute the convergence parameter**

$$Q = \frac{\max |E(\overset{\cap}{\omega}_i)| - \min |E(\overset{\cap}{\omega}_i)|}{\max |E(\overset{\cap}{\omega}_i)|}$$

   where $i = 0, 1, \ldots, \rho$.

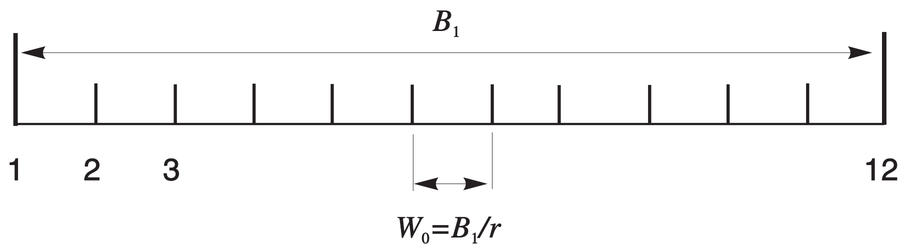
6. **Reject $\rho - r$ *superfluous potential extremals* $\overset{\cap}{\omega}_i$ according to an appropriate rejection criterion and renumber the remaining $\overset{\cap}{\omega}_i$ by setting $\hat{\omega}_i = \overset{\cap}{\omega}_i$ for $i = 0, 1, \ldots, r$.**

7. **If $Q > \varepsilon$, where ε is a convergence tolerance (say $\varepsilon = 0.01$), repeat from step 2; otherwise continue to step 8.**

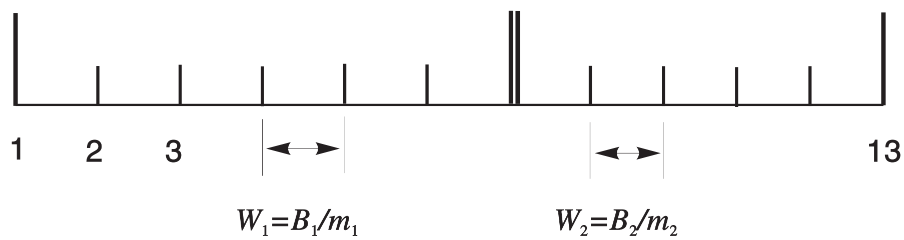
8. **Compute $P_c(\omega)$ using the last set of extremal frequencies; then deduce $h(n)$, the impulse response of the required filter, and stop.**

INITIALIZATION OF EXTREMAL FREQUENCIES

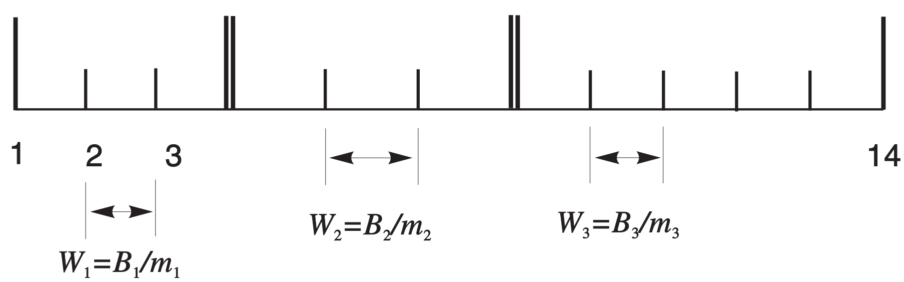
Bands: 1
Extremals: r+1 (12)
Intervals: r (11)



Bands: 2
Extremals: r+1 (13)
Intervals: r-1 (11)



Bands: 3
Extremals: r+1 (14)
Intervals: r-2 (11)



INITIALIZATION OF EXTREMAL FREQUENCIES *Cont'd*

For a filter with J bands with bandwidths $B_1,\, B_2,\, \ldots,\, B_J$, the number of extremals and interval between extremals for each band can be calculated by using the following formulas:

$$
\begin{aligned}
W_0 &= \frac{1}{r+1-J}\sum_{j=1}^{J} B_j \\
m_j &= \left(\frac{B_j}{W_0} + 0.5\right) \quad \text{for } j = 1,\, 2,\, \ldots,\, J-1 \\
&\qquad \text{and} \quad m_J = r - \sum_{j=1}^{J-1}(m_j + 1) \\
W_j &= \frac{B_j}{m_j} \quad \text{for } j = 1,\, 2,\, \ldots,\, J
\end{aligned}
$$

where

$$r = \frac{N+1}{2}$$

and N is the filter length.

UPDATING OF EXTREMALS

- **In each iteration the extremals need to be updated — this is done by finding the maxima of the error function**

$$|E(\omega)| = |W(\omega)[D(\omega) - P_c(\omega)]|$$

- **This could be done by solving the system**

$$\begin{bmatrix} 1 & \cos\hat{\omega}_0 & \cos\hat{\omega}_0 & \cdots & \cos\hat{\omega}_0 & \frac{1}{W(\hat{\omega}_0)} \\ 1 & \cos\hat{\omega}_1 & \cos\hat{\omega}_1 & \cdots & \cos\hat{\omega}_1 & \frac{1}{W(\hat{\omega}_1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \cos\hat{\omega}_r & \cos\hat{\omega}_r & \cdots & \cos\hat{\omega}_r & \frac{1}{W(\hat{\omega}_r)} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_c \\ \delta \end{bmatrix} = \begin{bmatrix} D(\hat{\omega}_0) \\ D(\hat{\omega}_1) \\ \vdots \\ D(\hat{\omega}_{r-1}) \\ D(\hat{\omega}_r) \end{bmatrix}$$

for the coefficients a_k and then calculating

$$P_c(\omega) = \sum_{k=0}^{c} a_k \cos k\omega$$

and in turn $E(\omega)$.

- **This approach is inefficient and may be subject to numerical ill-conditioning, in particular if δ is small and N is large.**

  Note that a 50×50 matrix is quite typical.

UPDATING OF EXTREMALS *Cont'd*

- **An alternative and more efficient approach is to deduce δ analytically (by using Cramer's rule) and then interpolate $P_c(\omega)$ on the r frequency points using the barycentric form of the Lagrange interpolation formula, as follows:**

- **Calculate parameter δ as**

$$\delta = \sum_{k=0}^{r} \frac{\alpha_k D(\hat{\omega}_k)}{\dfrac{\sum_{k=0}^{r} (-1)^k \alpha_k}{W(\hat{\omega})}}$$

- **With δ known, $P_c(\omega)$ can be obtained as**

$$P_c(\omega) = \begin{cases} C_k & \text{for } \omega = \hat{\omega}_0,\ \hat{\omega}_1,\ \ldots,\ \hat{\omega}_{r-1} \\ \dfrac{\displaystyle\sum_{k=0}^{r-1} \frac{\beta_k C_k}{x - x_k}}{\displaystyle\sum_{k=0}^{r-1} \frac{\beta_k}{x - x_k}} & \text{otherwise} \end{cases}$$

where

$$\begin{aligned} \alpha_k &= \prod_{i=0,\ i\neq k}^{r} \frac{1}{x_k - x_i} \\ \beta_k &= \prod_{i=0,\ i\neq k}^{r-1} \frac{1}{x_k - x_i} \\ C_k &= D(\hat{\omega}_k) - (-1)^k \frac{\delta}{W(\hat{\omega}_k)} \end{aligned}$$

with

$$x = \cos\omega \quad \text{and} \quad x_i = \cos\hat{\omega}_i \quad \text{for } i = 0,\ 1,\ \ldots,\ r$$

REJECTION OF SUPERFLUOUS POTENTIAL EXTREMALS

- **It follows from the alternation theorem that the minimized error function $|E(\omega)|$ has precisely $r+1$ extremals where $r=(N-1)/2$.**

  In addition, the problem formulation is such that there must be exactly $r+1$ extremals in each iteration.

- **Analysis will show that $|E(\omega)|$ can have as many as $r+2J-1$ maxima where J is the number of bands.**

  If in any iteration the number of maxima exceeds $r+1$, then the iteration is said to have generated *superfluous potential extremals*.

  For example, if in some iteration $\rho+1$ potential extremals are generated with $\rho>r$, then $\rho-r$ potential extremals must be rejected.

- **In the standard McClellan, Rabiner, and Parks algorithm, this difficulty is circumvented by rejecting the $\rho-r$ potential extremals $\overset{\cap}{\omega}_i$ that yield the lowest error $|E(\omega)|$.**

COMPUTATION OF IMPULSE RESPONSE

- **The impulse response in Step 8 of the algorithm can be determined by recalling that function $P_c(\omega)$ is the frequency response of a noncausal version of the required filter.**

- **The impulse response of the noncausal filter, denoted as $h_0(n)$ for $-c \le n \le c$, can be determined by computing $P_c(k\Omega)$ for $k = 0,\, 1,\, \ldots,\, c$ where $\Omega = 2\pi/N$, and then using the inverse discrete Fourier transform.**

- **It can be shown that**

$$h_0(n) = h_0(-n) = \frac{1}{N}\left\{P_c(0) + \sum_{k=1}^{c} 2P_c(k\Omega)\cos\left(\frac{2\pi kn}{N}\right)\right\}$$

  for $n = 0,\, 1,\, \ldots,\, c$.

- **The impulse response of the required causal filter is given by**

$$h(n) = h_0(n - c)$$

  for $n = 0,\, 1,\, \ldots,\, c$.

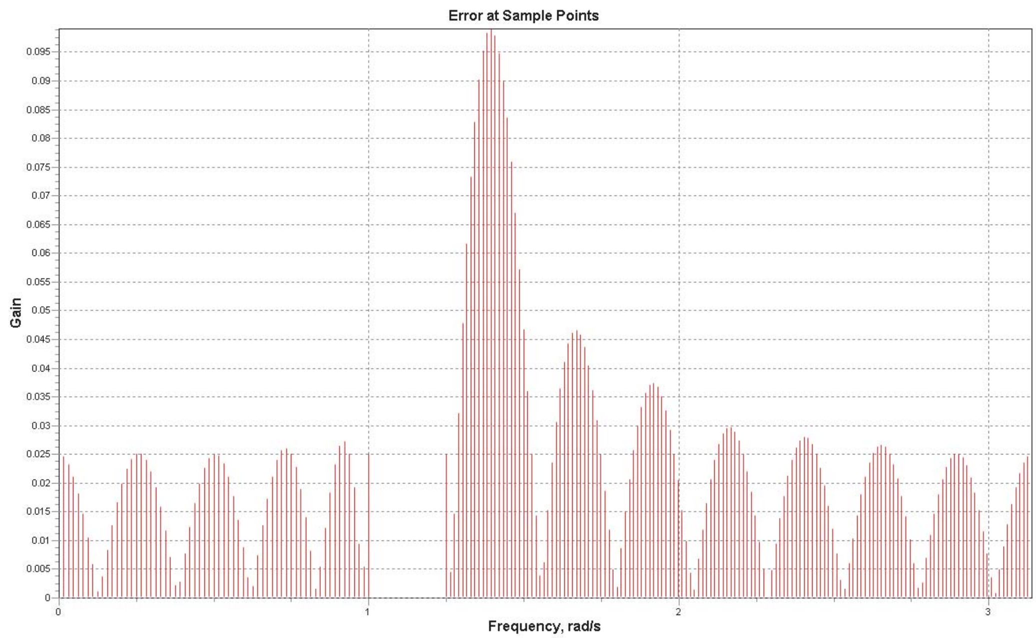
EXAMPLE

Band	$D(\omega)$	$W(\omega)$	Left band edge	Right band edge
1	1	1	0	1.0
2	0	0.4	1.25	π
Sampling frequency: 2π				

Filter length: 27
Iteration no: 3

Function Evals: 398

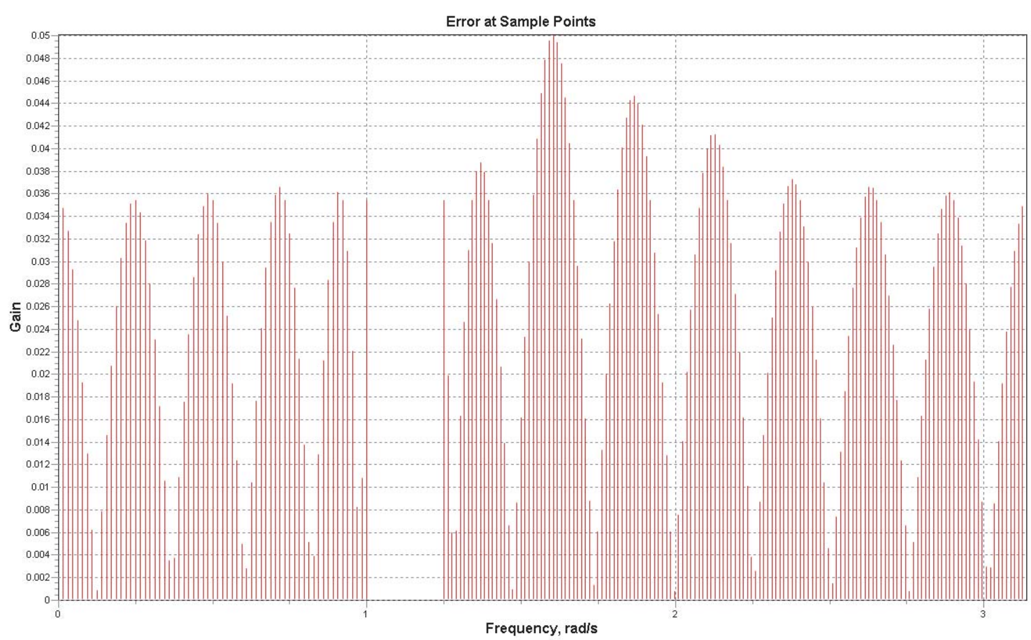
Iteration #3



Filter length: 27
Iteration no: 4

Function Evals: 597

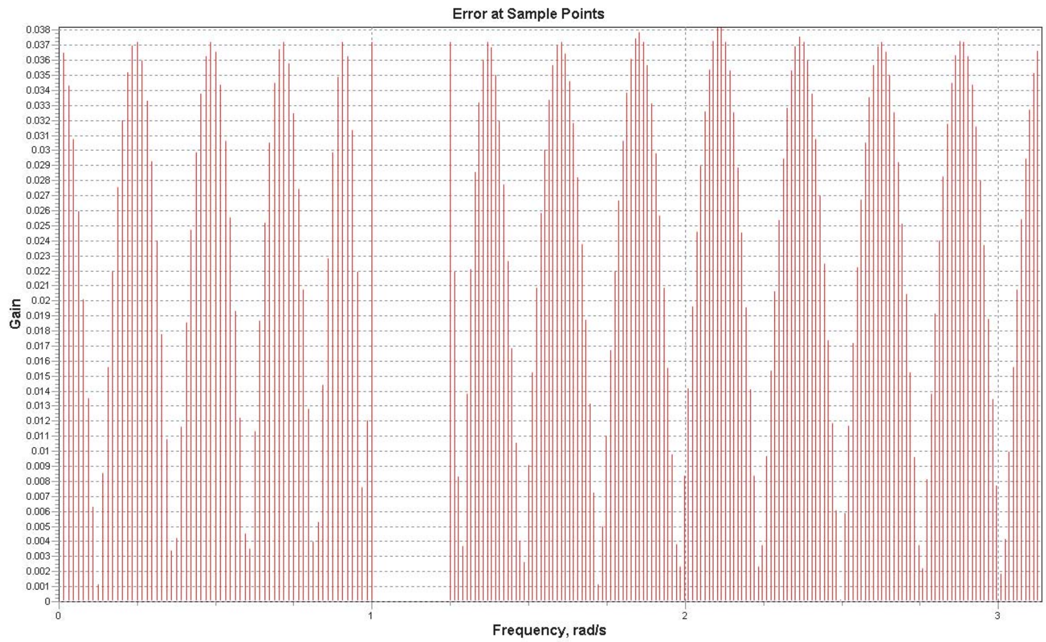
Iteration #4



Filter length: 27
Iteration no: 5

Function Evals: 796

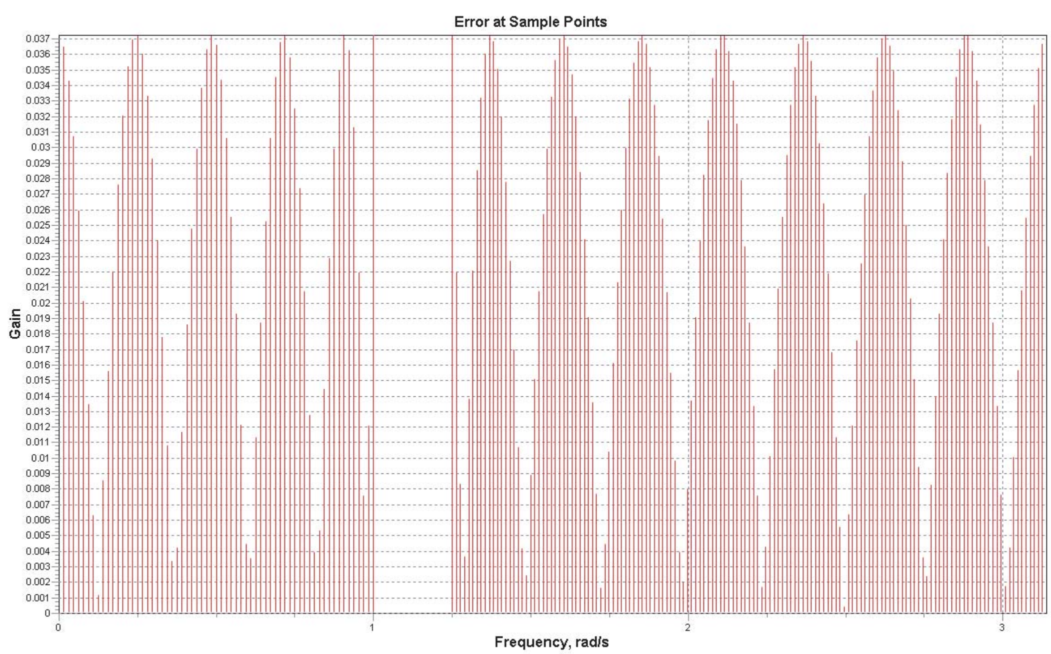
Iteration #5



Filter length: 27
Iteration no: 6

Function Evals: 995

Iteration #6



SELECTIVE STEP-BY-STEP SEARCH

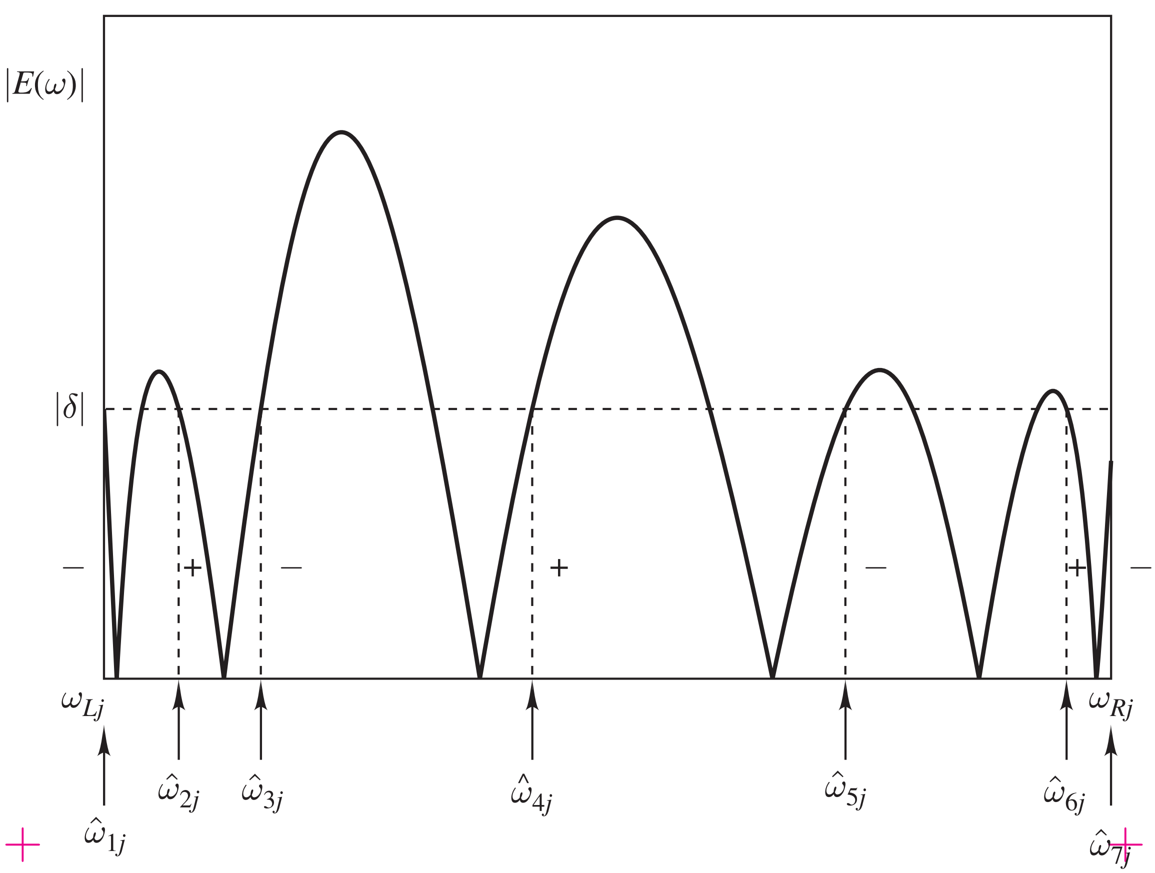
- **When the system of equations**

$$\begin{bmatrix} 1 & \cos\hat{\omega}_0 & \cos\hat{\omega}_0 & \cdots & \cos\hat{\omega}_0 & \frac{1}{W(\hat{\omega}_0)} \\ 1 & \cos\hat{\omega}_1 & \cos\hat{\omega}_1 & \cdots & \cos\hat{\omega}_1 & \frac{1}{W(\hat{\omega}_1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \cos\hat{\omega}_r & \cos\hat{\omega}_r & \cdots & \cos\hat{\omega}_r & \frac{1}{W(\hat{\omega}_r)} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_c \\ \delta \end{bmatrix} = \begin{bmatrix} D(\hat{\omega}_0) \\ D(\hat{\omega}_1) \\ \vdots \\ D(\hat{\omega}_{r-1}) \\ D(\hat{\omega}_r) \end{bmatrix}$$

is solved, the error function $|E(\omega)|$ is forced to satisfy the relation

$$|E(\hat{\omega}_i)| = |W(\hat{\omega}_i)[D(\hat{\omega}_i) - P_c(\hat{\omega}_i)]| = |\delta|$$

- **This relation can be satisfied in a number of ways but the most likely possibility for the jth band is illustrated below where $\omega_L j$ and $\omega_R j$ are the left-hand and right-hand edges, respectively.**



SELECTIVE STEP-BY-STEP SEARCH *Cont'd*

Because of the special nature of the error function

(a) **the maxima of $|E(\omega)|$ can be easily found by searching in the vicinity of the extremals;**

(b) **gradient information can be used to expedite the search for the maxima of $|E(\omega)|$; and**

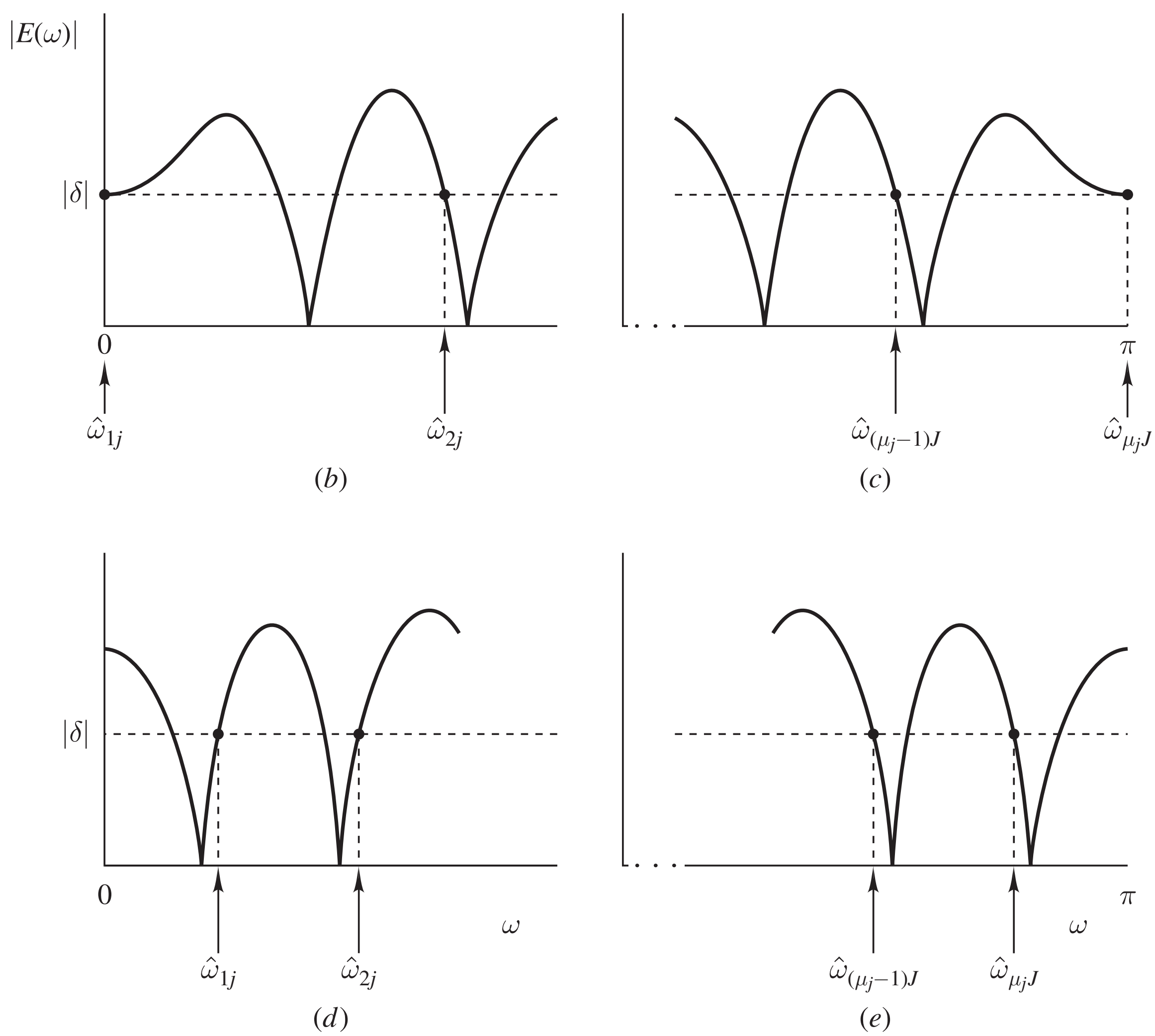
(c) **the closer we get to the solution, the closer are the maxima of the error function to the extremals.**

Thus by using a *selective step-by-step search*, a large amount of computation can be eliminated.

SELECTIVE STEP-BY-STEP SEARCH *Cont'd*

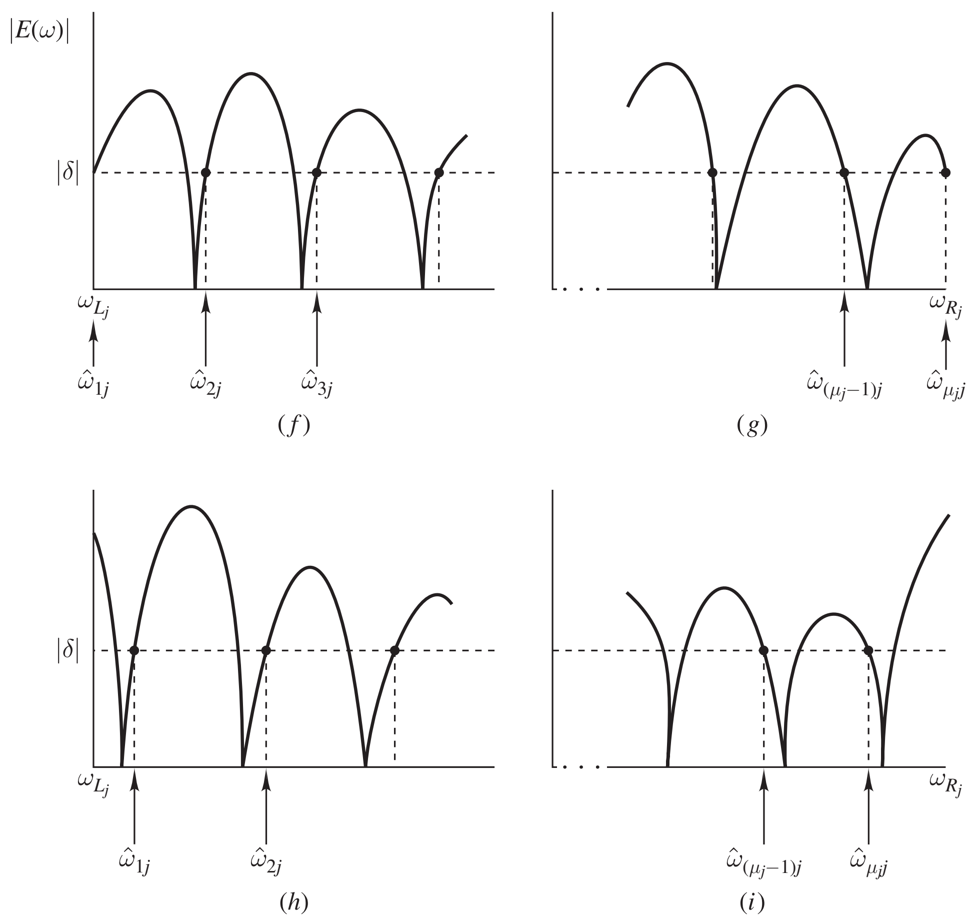
Unfortunately, extra ripples can arise at the band edges, as shown:

First and Last bands:



SELECTIVE STEP-BY-STEP SEARCH

Interior bands:



CUBIC INTERPOLATION

Increased computational efficiency can be achieved by using a search based on cubic interpolation.

Assuming that the error function shown in the figure can be represented by the third-order polynomial

$$|E(\omega)| = M = a + b\omega + c\omega^2 + d\omega^3$$

where $a,\ b,\ c$, and d are constants then

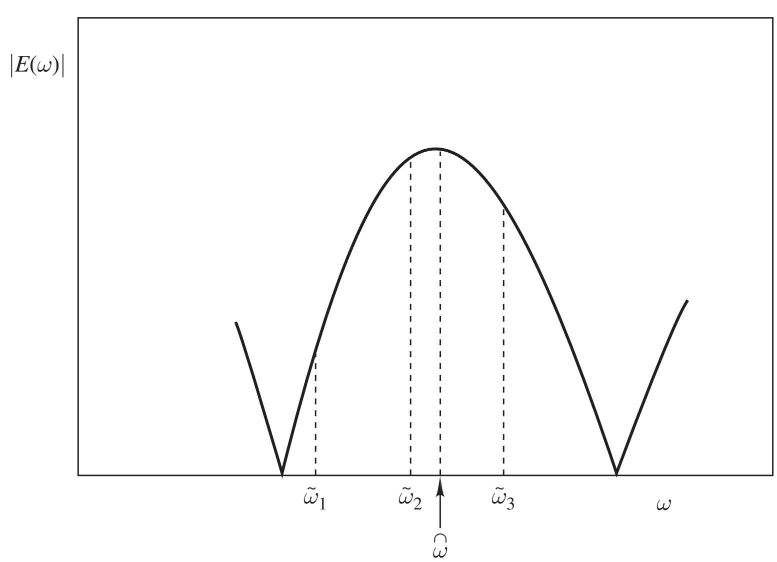
$$\frac{dM}{d\omega} = G = b + 2c\omega + 3d\omega^2$$

Hence, the frequencies at which M has stationary points are given by

$$\bar{\omega} = \frac{1}{3d}\left[-c \pm \sqrt{(c^2 - 3bd)}\right]$$

Therefore, $|E(\omega)|$ has a maximum if

$$\frac{d^2M}{d\omega^2} = 2c + 6d\overset{\frown}{\omega} < 0 \quad \text{or} \quad \overset{\frown}{\omega} < -\frac{c}{3d}$$



CUBIC INTERPOLATION *Cont'd*

- **The cubic interpolation method requires four function evaluations per potential extremal consistently.**
- **The selective step-by-step search may require as many as 8 function evaluations per potential extremal in the first two or three iterations but as the solution is approached only two or three function evaluations are required.**
- **By using the cubic interpolation to start with and then switching over to the step-by-step search, an very efficient algorithm can be constructed.**
- **The decision to switch from cubic to selective can be based on the value of the convergence parameter Q (see Step 5).**

  Switching from the cubic to the selective when Q is reduced below 0.65 works well.

IMPROVED REJECTION SCHEME FOR SUPERFLUOUS POTENTIAL EXTREMALS

- **If an extremal does not move from one iteration to the next, then the minimum value of $E(\overset{\cap}{\omega}_i)$ is simply δ, as can be easily shown, and this happens quite often even in the first or second iteration of the Remez algorithm.**

  As a consequence, rejecting potential extremals on the basis of the individual values of $E(\overset{\cap}{\omega}_i)$ tends to become random and this can slow the Remez algorithm quite significantly particularly for multiband filters.

- **An improved scheme for the rejection of superfluous extremals based the rejection on the lowest average band error as well as the individual values of $E(\overset{\cap}{\omega}_i)$ is described in the next transparency.**

IMPROVED REJECTION SCHEME FOR SUPERFLUOUS POTENTIAL EXTREMALS *Cont'd*

1. **Compute the average band errors**

$$E_j = \frac{1}{\nu_j} \sum_{\overset{\frown}{\omega}_i \in \Omega_j} |E(\overset{\frown}{\omega}_i)| \quad \text{for } j = 1,\, 2,\, \ldots,\, J$$

where Ω_j is the set of extremals in band j given by

$$\Omega_j = \{\overset{\frown}{\omega}_i : \omega_{Lj} \le \overset{\frown}{\omega}_i \le \omega_{Rj}\}$$

ν_j is the number of potential extremals in band j, and J is the number of bands.

2. **Rank the J bands in the order of lowest average error and let $l_1,\, l_2,\, \ldots,\, l_J$ be the ranked list obtained, i.e., l_1 and l_J are the bands with the lowest and highest average error, respectively.**

3. **Reject one $\overset{\frown}{\omega}_i$ in each of bands $l_1,\ l_2,\ \ldots,\ l_{J-1},\ l_1,\ l_2,$ ... until $\rho - r$ superfluous $\overset{\frown}{\omega}_i$ are rejected.**

   In each case, reject the $\overset{\frown}{\omega}_i$, other than a band edge, that yields the lowest $|E(\overset{\frown}{\omega}_i)|$ in the band.

EXAMPLE

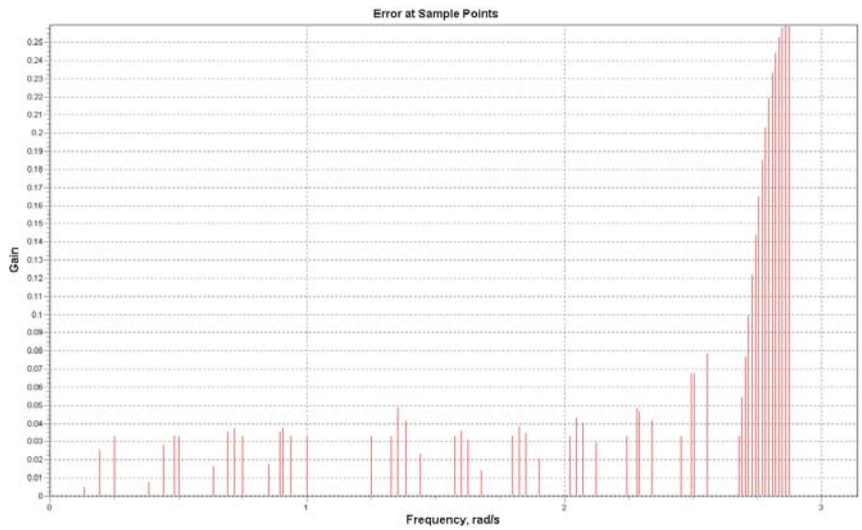
If $J = 3$, $\rho - r = 3$, and the average errors for bands 1, 2, and 3 are 0.05, 0.08, and 0.02, then $\overset{\frown}{\omega}_i$ are rejected in bands 3, 1, and 3.

Note that potential extremals are not rejected in band 2 which is the band of highest average error.

Filter length: 27
Iteration no: 3

Function Evals: 184

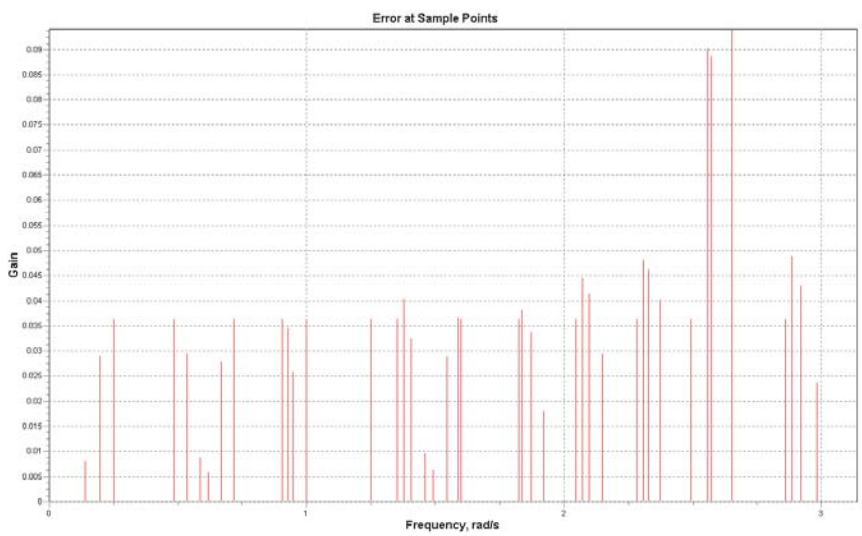
Iteration #3



Filter length: 27
Iteration no: 4

Function Evals: 186

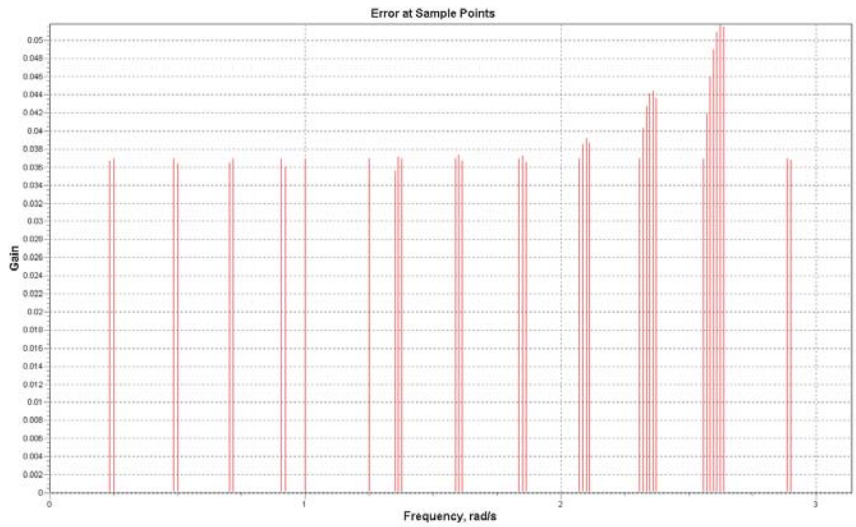
Iteration #4



Filter length: 27
Iteration no: 5

Function Evals: 223

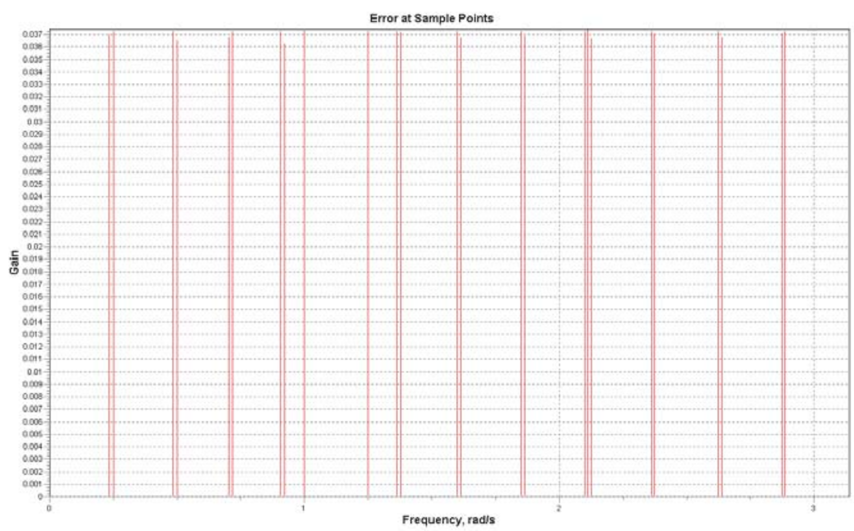
Iteration #5



Filter length: 27
Iteration no: 6

Function Evals: 263

Iteration #6



COMPARISONS — Amount of Computation

Type of Filter	No. of Examples	Range of N	Ave. Funct. Evals.			Saving, %	
			A	B	C	C versus B	C versus A
LP	45	9-101	2691	722	372	48.9	86.3
HP	42	9-101	2774	710	356	49.9	87.2
BP	44	21-89	2777	667	338	49.3	87.8
BS	35	21-91	2720	639	336	47.4	87.6

A: Exhaustive search

B: Selective search

C: Selective plus cubic search

COMPARISONS — Robustness

Type of Filter	No. of Examples	No. Failures		
		A	B	C
LP	46	1	0	0
HP	43	1	0	0
BP	50	3	2	5
BS	45	6	8	8

A: Exhaustive search
B: Selective search
C: Selective plus cubic search

CONCLUSIONS

- Three techniques that bring about substantial improvements in the efficiency of the Remez algorithm have been described:
  - A step-by-step exhaustive search
  - A cubic interpolation search
  - An improved scheme for the rejection of superfluous potential extremals
- Extensive experimentation has shown that the selective and cubic interpolation searches reduce the amount of computation required by the Remez algorithm by almost 90% without degrading its robustness.

  The rejection scheme described increases the efficiency and robustness of the Remez algorithm further but the scheme has not been compared with the original method of McClellan, Rabiner, and Parks.
- For off-line applications, the Remez algorithm continues to be the method of choice for the design of linear-phase filters, multiband filters, differentiators, Hilbert transformers.
- Despite the improvements described, the Remez continues to require a large amount of computation and further research needs to be undertaken to make it suitable for applications where the design has to be done in real or quasi-real time.
- More work needs to be done on the design of systems with complex coefficients, on the design of approximately linear-phase filters, on the application of the Remez algorithm for the design of recursive (IIR) filters, and also to reduce the amount of computation further.

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